

# Dark soliton in a disorder potential

Małgorzata Mochol, Marcin Płodzień, and Krzysztof Sacha

*Instytut Fizyki imienia Mariana Smoluchowskiego and Mark Kac Complex Systems Research Center,  
Uniwersytet Jagielloński, ulica Reymonta 4, PL-30-059 Kraków, Poland*

(Dated: February 10, 2012)

We consider a dark soliton in a Bose-Einstein condensate in the presence of a weak disorder potential. Deformation of the soliton shape is analyzed within the Bogoliubov approach and by employing an expansion in eigenstates of the Pöschl-Teller Hamiltonian. Comparison of the results with the numerical simulations indicates that the linear response analysis reveals a good agreement even if the strength of disorder is of the order of the chemical potential of the system. In the second part of the paper we concentrate on the quantum nature of the dark soliton and demonstrate that the soliton may reveal Anderson localization in the presence of disorder. The Anderson localized soliton may decay due to quasi-particle excitations induced by the disorder. However, we show that the corresponding lifetime is much longer than the condensate lifetime in a typical experiment.

PACS numbers: 03.75.Lm, 72.15.Rn, 05.30.Jp

## I. INTRODUCTION

Ultra-cold atomic gases become a playground where complex systems of solid state physics, nonlinear quantum optics or even cosmology can be efficiently simulated and investigated [1, 2]. The level of experimental control and detection is unprecedented and allows one to build quantum simulators, i.e. experimentally controlled systems that are able to mimic other systems difficult to investigate directly [3]. Since the first laboratory achievement of Bose-Einstein condensation the list of problems investigated experimentally and theoretically in ultra-cold atomic gases becomes very long and includes: superfluid phases of both bosonic and fermionic atomic species [4, 5], collective excitations like solitons and vortices [6–12], insulating phases of solid state physics [13–15], transport properties and Anderson localization effects [16–19], atomic systems in the presence of artificial gauge potentials [20–22].

Particularly interesting is the interplay between particle interactions and disorder phenomena in quantum many body systems. Anderson localization [23], which is essentially a single particle phenomenon, is vulnerable to particle interactions [24–29]. In experiments with atomic gases the localization was observed when the interactions were practically turned off by employing Feshbach resonances or by reducing density of the atomic cloud [16–19].

Particle interactions play a vital role in soliton formation. Mean field description of atomic Bose-Einstein condensates reduces to the Gross-Pitaevskii equation [30–32] which possesses a localized wave-packet solution (i.e. bright soliton) for attractive particle interactions and a solution with a phase flip (i.e. dark soliton) for repulsive interactions. Both kinds of the solitons have been observed experimentally [6–9].

While the Gross-Pitaevskii equation is a single particle description with the interactions included in the mean field approximation one can anticipate quantum many body effects that go beyond such a description. Center of mass of a bright soliton is a degree of freedom

which, when properly described quantum mechanically, allows for the analysis of interesting phenomena like soliton scattering on a potential barrier which leads to a superposition of macroscopically distinct objects [33] or quantum entanglement of a pair of solitons [34]. It is also shown that in spite of the fact that the particle interactions are present and are responsible for the bright soliton formation the center of mass of the soliton can reveal Anderson localization in a weak disorder potential [35, 36]. One may raise a natural question if similar localization can be also observed in the case of a dark soliton. In the bright soliton case a huge energy gap for quasi-particle excitations guarantees that the shape of the soliton is not perturbed by a weak disorder potential. Then, the only degree of freedom affected by the disorder is the center of mass and (as has been shown recently) it becomes Anderson localized. In the dark soliton case the situation is more delicate because there is practically no gap for quasi-particle excitations. Consequently one may expect that coupling between the degree of freedom that describes soliton position and the quasi-particle subsystem induced by the disorder can destroy Anderson localization of the dark soliton. In the present paper we address this problem and show that the dark soliton localization can be observed experimentally because the lifetime of the localized states is sufficiently long.

The paper is organized as follows. In Sec. II we analyze the deformation of a dark soliton solution in the presence of a weak external potential within the Bogoliubov approach and by reducing the description to the problem of the Pöschl-Teller potential. Such a classical description of the soliton allows us to conclude that the deformation effect is very weak. It implies that in the quantum description of the soliton we may expect that coupling between the degree of freedom that describes soliton position and the quasi-particle subsystem can be neglected. It is confirmed in Sec. III where we switch to the quantum description and calculate the lifetimes of Anderson localized dark solitons. We conclude in Sec. IV.

## II. CLASSICAL DESCRIPTION

We consider  $N_0$  bosonic atoms with repulsive interactions in a 1D box potential of length  $L$  at zero temperature (for experimental realization of a box potential see [37]). The mean field description assumes that all atoms occupy the same single particle state  $\phi_0$  which is a solution of the Gross-Pitaevskii equation (GPE) [30]

$$-\frac{\hbar^2}{2m}\partial_x^2\phi_0 + g_0|\phi_0|^2\phi_0 = \mu_0\phi_0, \quad (1)$$

where  $g_0 = 2\hbar a\omega_\perp$ ,  $\mu_0$  is chemical potential of the system,  $a$  stands for the  $s$ -wave scattering length of the atoms and  $\omega_\perp$  denotes the transverse harmonic confinement frequency. By virtue of the nonlinear term in the GPE there exists a stationary dark soliton solution which, far from the boundary of the box potential, takes the form

$$\phi_0(x - q) = e^{-i\theta} \sqrt{\rho_0} \tanh\left(\frac{x - q}{\xi}\right), \quad (2)$$

where  $\theta$  is an arbitrary phase,  $\rho_0$  stands for the atomic density away from the soliton position  $q$  and  $\xi = \hbar/\sqrt{m g_0 \rho_0}$  is the so-called healing length. The chemical potential of the system is  $\mu_0 = g_0 \rho_0$ . The description of a dark soliton restricted to the classical wave equation (1) is called the classical description. It is in contrast to the quantum description of Sec. III where, for example, the position of a dark soliton  $q$  becomes a quantum degree of freedom. We consider a finite 1D system. However, in order to describe the system in a region away from the boundary of the box potential we may use analytical solutions corresponding to the infinite configuration space which, if necessary, can be modified to satisfy boundary conditions [38], e.g.  $\phi_0(x = 0) = 0$  and  $\phi_0(x = L) = 0$ .

In the remaining part of the paper we adapt the following units for energy, length and time, respectively

$$\begin{aligned} E_0 &= \mu_0, \\ l_0 &= \xi, \\ t_0 &= \frac{\hbar}{\mu_0}. \end{aligned} \quad (3)$$

### A. Deformation of a dark soliton: Expansion in Bogoliubov modes

We would like to describe the deformation of a dark soliton solution in the presence of a weak disorder potential. The disorder potentials we are interested in correspond to optical speckle potentials which are created experimentally by shining laser radiation on a so-called diffusive plate [28, 39]. In the far field, light forms a random intensity pattern which is experienced by atoms as an external disorder potential. Diffraction effects are responsible for a finite correlation length of the speckle potentials. Interestingly properties of such a disorder can be easily engineered that allows, e.g., for preparation of a matter-wave analog of an optical random laser [40].

Bright soliton deformation in an optical speckle potential has been already considered in Ref. [41], see also [42]. Dark soliton propagation and its radiation in the presence of randomly distributed Dirac-delta potentials has been considered in Ref. [43], see also [44, 45]. In the case of repulsive particle interactions the deformation of a condensate ground state in an optical speckle potential has been also a subject of scientific publications [46–49]. In the present paper we consider a problem of the repulsive interactions but we deal with a localized soliton structure, i.e. there is a phase flip of the condensate wavefunction at the position of a soliton.

In our considerations the dark soliton is placed in a bounded and weak external potential  $V(x)$ . To calculate a small perturbation of the solitonic wavefunction we start with the time-independent GPE which in our units (3) takes the form

$$-\frac{1}{2}\partial_x^2\phi(x) + \frac{1}{\rho_0}|\phi(x)|^2\phi(x) + V(x)\phi(x) = \mu\phi(x), \quad (4)$$

and substitute  $\phi = \phi_0 + \delta\phi$  and  $\mu = \mu_0 + \delta\mu = 1 + \delta\mu$  where  $\delta\phi$  is a small perturbation of the soliton wavefunction (2) and  $\delta\mu$  is a small contribution to the chemical potential which allows us to correct a possible change of the total particle number due to the presence of  $V(x)$ . Keeping linear terms only we obtain the time-independent, non-homogeneous Bogoliubov–de Gennes equations

$$\mathcal{L} \begin{bmatrix} \delta\phi \\ \delta\phi^* \end{bmatrix} = V \begin{bmatrix} -\phi_0 \\ \phi_0^* \end{bmatrix} + \delta\mu \begin{bmatrix} \phi_0 \\ -\phi_0^* \end{bmatrix}, \quad (5)$$

where

$$\mathcal{L} = \begin{pmatrix} -\frac{1}{2}\partial_x^2 + \frac{2}{\rho_0}|\phi_0|^2 - 1 & +\frac{1}{\rho_0}\phi_0^2 \\ -\frac{1}{\rho_0}\phi_0^{*2} & \frac{1}{2}\partial_x^2 - \frac{2}{\rho_0}|\phi_0|^2 + 1 \end{pmatrix}. \quad (6)$$

In order to solve Eqs. (5) we would like to expand the two component vector  $(\delta\phi, \delta\phi^*)^T$  in a complete basis that consists of eigenvectors of the non-Hermitian operator  $\mathcal{L}$ . This basis has been published in Ref. [38], see also [50, 51], and here we only present the results and comment on essential elements of the derivation. Collecting the complete basis one has to be careful because the  $\mathcal{L}$  operator is not diagonalizable [35, 38, 50, 51]. There are right eigenstates of  $\mathcal{L}$ , i.e.  $\mathcal{L}|\psi_k\rangle = \epsilon_k|\psi_k\rangle$ , where  $|\psi_k\rangle = (|u_k\rangle, |v_k\rangle)^T$  and [38]

$$u_k(x) = \frac{e^{ikx}e^{-i\theta}}{4\sqrt{\pi}\epsilon_k^{3/2}} \left[ (k^2 + 2\epsilon_k) \left( \frac{k}{2} + i \tanh(x - q) \right) + \frac{k}{\cosh^2(x - q)} \right], \quad (7)$$

$$v_k(x) = \frac{e^{ikx}e^{i\theta}}{4\sqrt{\pi}\epsilon_k^{3/2}} \left[ (k^2 - 2\epsilon_k) \left( \frac{k}{2} + i \tanh(x - q) \right) + \frac{k}{\cosh^2(x - q)} \right], \quad (8)$$

that correspond to the familiar Bogoliubov spectrum

$$\epsilon_k = \frac{1}{2}\sqrt{4k^2 + k^4}. \quad (9)$$

These eigenstates are called phonons and their adjoint vectors  $|\psi_k^{ad}\rangle = (|u_k\rangle, -|v_k\rangle)^T$  are also the left eigenmodes of  $\mathcal{L}$ . Due to symmetries of the  $\mathcal{L}$  operator  $|\tilde{\psi}_k\rangle = (|\tilde{u}_k\rangle, |\tilde{v}_k\rangle)^T = (|v_k^*\rangle, |u_k^*\rangle)^T$  are the right eigenmodes corresponding to eigenvalues  $\tilde{\epsilon}_k = -\epsilon_k$  and  $|\tilde{\psi}_k^{ad}\rangle = (-|v_k^*\rangle, |u_k^*\rangle)^T$  are their adjoint vectors. In the infinite configuration space right eigenvectors and adjoint vectors fulfill  $\langle\psi_k^{ad}|\psi_{k'}\rangle = \langle u_k|u_{k'}\rangle - \langle v_k|v_{k'}\rangle = \delta(k-k')$ . In a box potential the wavevector becomes quantized, i.e.  $k_n = n\pi/L$  where  $n = 1, 2, \dots$ , and there is a tiny energy gap for the phonon excitations.

Phonons apart, there exist two zero-eigenvalue modes of  $\mathcal{L}$  corresponding to broken gauge  $U(1)$  and translational symmetries [38],

$$\begin{bmatrix} u_\theta \\ v_\theta \end{bmatrix} = i \frac{\partial}{\partial \theta} \begin{bmatrix} \phi_0 \\ \phi_0^* \end{bmatrix}, \quad (10)$$

$$\begin{bmatrix} u_q \\ v_q \end{bmatrix} = i \frac{\partial}{\partial q} \begin{bmatrix} \phi_0 \\ \phi_0^* \end{bmatrix}, \quad (11)$$

respectively. They appear as zero-eigenvalue vectors because a shift of the global phase  $\theta$  or a change of the soliton position  $q$  in solution (2) does not lead to a change of the system energy. The zero modes fulfill  $\langle u_q|u_\theta\rangle - \langle v_q|v_\theta\rangle = 0$  and they are orthogonal to the modes adjoint to the phonons. Vectors adjoint to the zero modes are not left eigenvectors of the  $\mathcal{L}$  and they can be found by solving [51]

$$\mathcal{L} \begin{bmatrix} u_{\theta,q}^{ad} \\ v_{\theta,q}^{ad} \end{bmatrix} = \frac{1}{M_{\theta,q}} \begin{bmatrix} u_{\theta,q} \\ v_{\theta,q} \end{bmatrix}, \quad (12)$$

where  $M_{\theta,q}$  are determined by normalization conditions  $\langle u_{\theta,q}^{ad}|u_{\theta,q}\rangle - \langle v_{\theta,q}^{ad}|v_{\theta,q}\rangle = 1$ . One gets [38]

$$\begin{bmatrix} u_{\theta,q}^{ad} \\ v_{\theta,q}^{ad} \end{bmatrix} = \frac{\partial}{\partial N_0} \begin{bmatrix} \phi_0 \\ \phi_0^* \end{bmatrix} - iR \begin{bmatrix} u_q \\ v_q \end{bmatrix}, \quad (13)$$

$$\begin{bmatrix} u_q^{ad} \\ v_q^{ad} \end{bmatrix} = -\frac{i}{4\sqrt{\rho_0}} \begin{bmatrix} e^{-i\theta} \\ -e^{i\theta} \end{bmatrix}, \quad (14)$$

where  $M_\theta = \rho_0(\partial N_0/\partial \rho_0)$ ,  $M_q = -4\rho_0$  and  $R = (2q - L)\rho_0/M_q M_\theta$ . We have  $\langle u_q^{ad}|u_\theta\rangle - \langle v_q^{ad}|v_\theta\rangle = 0$  and  $\langle u_\theta^{ad}|u_q\rangle - \langle v_\theta^{ad}|v_q\rangle = 0$ . A small contribution of the zero mode  $(u_q, v_q)^T$  in (13) allows one to fulfill also  $\langle u_q^{ad}|u_\theta^{ad}\rangle - \langle v_q^{ad}|v_\theta^{ad}\rangle = 0$ .

Now we have all vectors to build a complete basis and the deformation of the soliton can be expanded in that basis

$$\begin{bmatrix} \delta\phi \\ \delta\phi^* \end{bmatrix} = \Delta\theta \begin{bmatrix} u_\theta \\ v_\theta \end{bmatrix} + P_\theta \begin{bmatrix} u_\theta^{ad} \\ v_\theta^{ad} \end{bmatrix} + \Delta q \begin{bmatrix} u_q \\ v_q \end{bmatrix} + P_q \begin{bmatrix} u_q^{ad} \\ v_q^{ad} \end{bmatrix} + \sum_k \left( b_k \begin{bmatrix} u_k \\ v_k \end{bmatrix} + b_k^* \begin{bmatrix} v_k^* \\ u_k^* \end{bmatrix} \right). \quad (15)$$

Substituting (15) into (5) results in

$$V \begin{bmatrix} -\phi_0 \\ \phi_0^* \end{bmatrix} + \delta\mu \begin{bmatrix} \phi_0 \\ -\phi_0^* \end{bmatrix} = \frac{P_\theta}{M_\theta} \begin{bmatrix} u_\theta \\ v_\theta \end{bmatrix} + \frac{P_q}{M_q} \begin{bmatrix} u_q \\ v_q \end{bmatrix}$$

$$+ \sum_k \epsilon_k \left( b_k \begin{bmatrix} u_k \\ v_k \end{bmatrix} - b_k^* \begin{bmatrix} v_k^* \\ u_k^* \end{bmatrix} \right). \quad (16)$$

Projecting this equation onto the adjoint vectors we can obtain the expansion coefficients and a small correction to the chemical potential. Note that, there is no restriction for a choice of small deviation  $\Delta\theta$  and  $\Delta q$  which is due to the fact that these coefficients are related to the zero modes. However, while  $\theta$  can be arbitrary, we will see that  $q$  can not because the external potential breaks the translational symmetry but does not affect the  $U(1)$  symmetry. Coefficient  $\Delta\theta$  is related to a shift of the global phase of the soliton solution (2) and without loss of generality we may choose  $\Delta\theta = 0$ . For the coefficient  $P_\theta$  we get

$$\begin{aligned} \frac{P_\theta}{M_\theta} &= -2\langle\partial_{N_0}\phi_0|V\phi_0\rangle + \delta\mu \\ &\quad - iR(\langle u_q|V\phi_0\rangle + \langle v_q|V\phi_0^*\rangle). \end{aligned} \quad (17)$$

The last term on the r.h.s. can be written in the following way

$$\langle u_q|V\phi_0\rangle + \langle v_q|V\phi_0^*\rangle \sim \int_0^L dx |\phi_0(x-q)|^2 \partial_x V(x), \quad (18)$$

which represents a force acting on the soliton. A similar effective force appears in the approach of Ref. [52]. For the stationary solution it is obvious that the soliton position  $q$  is chosen so that such a force is zero. Then, also an arbitrary shift of the soliton position should be zero, i.e.  $\Delta q = 0$  in (15). With such a choice of the soliton position the coefficient  $P_\theta$  is directly related to a change of the total particle number which we assume to be zero. Hence,  $P_\theta = dN = 0$  and Eq. (17) allows us to obtain the correction to the chemical potential

$$\begin{aligned} \delta\mu &= 2\langle\partial_{N_0}\phi_0|V\phi_0\rangle \\ &= \frac{1}{L} \int_0^L dy (\tanh y + y \operatorname{sech}^2 y) \tanh y V(y+q). \end{aligned} \quad (19)$$

Projecting (16) onto the adjoint mode (14) results in  $P_q = 0$  what can be expected because  $P_q$  has an interpretation of the soliton momentum and for the stationary state it should be zero.

Thus, with a proper choice of the soliton position and a suitable correction of the chemical potential, all coefficients in (15) are zero except

$$b_k = \frac{1}{\epsilon_k} [-\langle u_k|V\phi_0\rangle - \langle v_k|V\phi_0^*\rangle], \quad (20)$$

which contain full information about soliton deformation in a weak external potential. Finally the stationary solitonic solution in the presence of a weak external potential reads

$$\phi(x) = \phi_0(x) + \sum_k [b_k u_k(x) + b_k^* v_k^*(x)]. \quad (21)$$

### B. Deformation of a dark soliton: Expansion in modes of the Pöschl–Teller potential

The Bogoliubov approach is suitable in the description of the eigenmodes of collective or elementary excitations of a Bose-Einstein condensate [31, 32]. However, if we are interested in the description of a stationary state of the GPE and if such a state can be represented as a real function one may introduce a substantial simplification. We will see that description of dark soliton deformation reduces to an expansion of a wavefunction perturbation in modes of the Pöschl–Teller potential [53]. Such an approach has been applied in the analysis of a bright soliton deformation in a weak disorder potential [41] and it turns out it is also suitable in the dark soliton case.

Let us start again with the stationary GPE but assume the solution we are looking for is a real function

$$\left(-\frac{1}{2}\partial_x^2 + \frac{1}{\rho_0}\phi^2 - \mu + V(x)\right)\phi = 0. \quad (22)$$

Similarly to the Bogoliubov approach we introduce  $\mu = \mu_0 + \delta\mu = 1 + \delta\mu$  and  $\phi = \phi_0 + \delta\phi$  but assume that in (2) the global phase  $\theta = 0$ . Keeping linear terms only, rewriting  $\phi_0^2(x - q) = \rho_0 \tanh^2(x - q) = \rho_0 [1 - \cosh^{-2}(x - q)]$  and changing variable  $x \rightarrow x + q$  we obtain

$$(H_0 + 2)\delta\phi = \delta\mu\phi_0 - V(x + q)\phi_0, \quad (23)$$

where

$$H_0 = -\frac{1}{2}\partial_x^2 - \frac{3}{\cosh^2(x)}, \quad (24)$$

is the Hamiltonian for a particle in the Pöschl–Teller potential [53]. To compute  $\delta\phi$  we need to invert the operator  $H_0 + 2$ . All the eigenstates of the Hamiltonian  $H_0$  are known in the literature [54]. There are two bound states

$$\psi_0(x) = \frac{\sqrt{3}}{2}\text{sech}(x)^2, \quad (25)$$

$$\psi_1(x) = \sqrt{\frac{3}{2}}\text{sech}(x)\tanh(x), \quad (26)$$

with eigenenergies  $E_0 = -2$  and  $E_1 = -\frac{1}{2}$  respectively and scattering states

$$\psi_k(x) = \frac{e^{ikx}}{(2\pi)^{1/2}} \frac{k^2 - 2 + 3\text{sech}(x)^2 + 3ik \tanh(x)}{[(1 + k^2)(4 + k^2)]^{1/2}}, \quad (27)$$

with  $E_k = \frac{k^2}{2}$ ,  $k \in \mathbb{R}$ . We can therefore expand the deformation  $\delta\phi$  over orthonormal basis of eigenfunctions

$$\delta\phi = \alpha_0 \psi_0 + \alpha_1 \psi_1 + \int dk \alpha_k \psi_k(x), \quad (28)$$

and compute coefficients  $\alpha_j$  by projecting Eq. (23) on the proper eigenmodes

$$(E_j + 2)\alpha_j = \int dx \psi_j^*(x)[\delta\mu\phi_0 - V(x + q)\phi_0]. \quad (29)$$

The wavefunction (25) is a zero mode, i.e.  $(H_0 + 2)\psi_0 = 0$ . Thus, Eq. (23) can be solved provided the projection of its r.h.s on the zero mode vanishes. Therefore we require

$$\begin{aligned} -\langle\psi_0|V\phi_0\rangle + \delta\mu\langle\psi_0|\phi_0\rangle &= -\langle\psi_0|V\phi_0\rangle \sim \langle\partial_x\phi_0|V\phi_0\rangle \\ &= -\int_0^L dx \phi_0^2(x - q)\partial_x V(x) \\ &= 0. \end{aligned} \quad (30)$$

In (30) we have taken advantage of  $\langle\psi_0|\phi_0\rangle = 0$  and the fact that the zero mode is also the translation mode of the dark soliton, i.e.  $\psi_0 \sim \partial_x\phi_0$ . Condition (30) implies that the soliton position  $q$  should be chosen so that the force acting on it is zero, compare with (18). From Eq. (29) we do not get any restriction for the value of  $\alpha_0$ . However, because  $\alpha_0\psi_0$  has an interpretation of a shift of the soliton position we should choose  $\alpha_0 = 0$  if we are interested in a stationary solution.

In order to solve Eq. (23) we have to invert the operator  $H_0 + 2$  in the Hilbert space with the zero mode excluded which is simple because all eigenfunctions of  $H_0$  are known. That is

$$\delta\phi(x) = \int dy K(x, y) [\delta\mu\phi_0 - V(y + q)\phi_0], \quad (31)$$

where the symmetric kernel  $K(x, y)$  reads

$$\begin{aligned} K(x, y) &= \frac{2}{3}\psi_1(x)\psi_1^*(y) + 2 \int \frac{\psi_k(x)\psi_k^*(y)}{4 + k^2} \\ &= -\frac{1}{16}\text{sech}^2(x)\text{sech}^2(y) \times \\ &\quad \{ \text{sh}^2 2x + \text{sh}^2 2y + 4\text{ch} 2x + 4\text{ch} 2y \\ &\quad - 3 - (\text{ch} 2x + \text{ch} 2y + 3) |\text{sh} 2x - \text{sh} 2y| \\ &\quad - 4\text{sh}|x - y|\text{sh} x \text{sh} y - 6|x - y| \}. \end{aligned} \quad (32)$$

In the Bogoliubov approach, even if we are restricted to the phonon subspace, the  $\mathcal{L}$  operator possesses a zero eigenvalue if the configuration space is infinite, see (9). Therefore it is not straightforward to obtain a simple form of the relevant kernel because integration over phonon subspace should actually be substituted by summation over discrete values of the wavevector.

Finally we have to determine the correction to the chemical potential. In the Bogoliubov approach there is a specific mode responsible for a change of the total particle number which is orthogonal to all other Bogoliubov modes and to keep the particle number constant it is sufficient to ensure that the wavefunction perturbation does not have any contribution from this mode. In the present approach the chemical potential has to be determined by the normalization condition  $\langle\phi|\phi\rangle = N_0 + \mathcal{O}(\delta\phi^2)$  that implies  $\langle\phi_0|\delta\phi\rangle = 0$  and thus

$$\begin{aligned} \delta\mu &= \frac{\int dx dy \phi_0(x)K(x, y)V(y + q)\phi_0(y)}{\int dx dy \phi_0(x)K(x, y)\phi_0(y)} \\ &= \frac{1}{L} \int dy (\tanh y + y \text{sech}^2 y) \tanh y V(q + y), \end{aligned}$$



(33)

which is the same expression as in the Bogoliubov approach, compare with (19).

The final expression for a stationary solitonic solution in the presence of a weak external potential reads

$$\begin{aligned} \phi(x) = & \phi_0(x) - \int dy K(x, y) V(y + q) \phi_0(y) \\ & + \delta\mu \frac{\partial \phi_0(x)|_{\mu_0}}{\partial \mu_0}, \end{aligned} \quad (34)$$

where we have made use of  $\int dy K(x, y) \phi_0(y) = \frac{\partial \phi_0(x)|_{\mu_0}}{\partial \mu_0}$  with  $\phi_0|_{\mu_0} \equiv \phi_0$  being a wave function with fixed chemical potential  $\mu_0 = 1$ .

In the bright soliton case [41] the eigenfunction  $\psi_1$  (not  $\psi_0$  as in the dark soliton problem) turns out to be a translation mode of the system and also the  $H_0 + 2$  operator is substituted by  $H_0 + 1/2$ . Consequently, one obtains a different expression for the symmetric kernel which is expected because the bright soliton represents a localized wavepacket while a dark soliton is a phase flip at soliton position with otherwise uniform density.

### C. Comparison with numerical calculations

We are going to compare the perturbative approaches introduced in the previous subsections with results of numerical calculations in the case of a dark soliton in a weak optical speckle potential. However, first we consider a simple harmonic potential

$$V(x) = -V_0 \cos(k_0 x), \quad (35)$$

which on one hand is a generic example because any potential can be expanded in a Fourier basis and on the other hand allows us to obtain simple expressions for expansion coefficients.

In the Bogoliubov approach Eq. (20), for the potential (35), one gets

$$\begin{aligned} b_k = & -iV_0 \sqrt{\rho_0} \frac{\sqrt{\pi}}{2k(4+k^2)\sqrt{\epsilon_k}} \times \\ & \left\{ k_0^2 \left( \operatorname{csch} \frac{\pi(k-k_0)}{2} + \operatorname{csch} \frac{\pi(k+k_0)}{2} \right) \right. \\ & \left. + 4k [\delta(k-k_0) + \delta(k+k_0)] \right\}. \end{aligned} \quad (36)$$

This coefficient contains Dirac delta functions which tell us that the condensate density will be modulated with a period of  $2\pi/k_0$ . At small  $k$  the coefficient diverges

$$b_k \sim \frac{1}{\sqrt{|k|}} + \mathcal{O}(|k|^{3/2}), \quad (37)$$

but the contribution to the soliton wavefunction from long wavelength phonons is finite

$$b_k u_k + b_k^* v_k^* \sim \text{const} + \mathcal{O}(k^2). \quad (38)$$

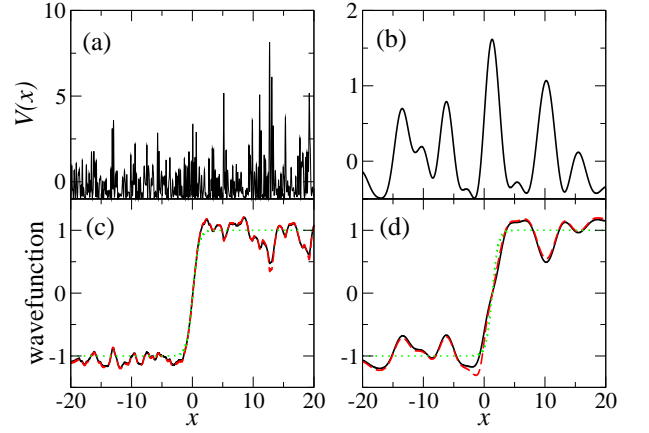


Figure 1: (Color online) In panel (a) we show an example of the optical speckle potential with the correlation length  $\sigma_R = 0.05$  and for  $V_0 = 1$  while in panel (c) we present the corresponding exact solution of the Gross-Pitaevskii equation obtained numerically (solid black line) and within the perturbation approach (red dashed line), see Eq.(34) (Eq.(21) leads to the same results). In panels (b) and (d) we show the same as in (a) and (c) but for  $\sigma_R = 1$  and  $V_0 = 0.5$ . Green dotted lines in (c) and (d) correspond to unperturbed soliton wavefunctions (2).

Thus, in order to calculate soliton deformation we may assume infinite configuration space because the singularity at  $k \rightarrow 0$  is actually not dangerous. The correction to the chemical potential  $|\delta\mu| < \frac{4V_0}{k_0 L}$ , see Eq. (19), disappears in the limit of  $k_0 L \rightarrow \infty$  which is a consequence of the self-averaging property of the potential (35), see [47].

In the approach that involves modes of the Pöschl-Teller potential Eq. (29) yields

$$\begin{aligned} \alpha_k = & iV_0 \sqrt{\rho_0} \frac{\sqrt{\pi}}{\sqrt{8}(1+k^2)^{1/2}(4+k^2)^{3/2}} \times \\ & \left\{ (k^2 - 3k_0^2 + 4) \left( \operatorname{csch} \frac{\pi(k-k_0)}{2} + \operatorname{csch} \frac{\pi(k+k_0)}{2} \right) \right. \\ & \left. - 12k [\delta(k+k_0) + \delta(k-k_0)] \right\}, \end{aligned} \quad (39)$$

for scattering modes and

$$\alpha_1 = V_0 \sqrt{\frac{2\rho_0}{3}} \frac{\pi}{2} (1 - k_0^2) \operatorname{sech} \left( \frac{k_0 \pi}{2} \right), \quad (40)$$

for the bound state  $\psi_1$ , where we have assumed that the system is infinite and thus the correction  $\delta\mu = 0$ . There are Dirac delta functions present in (39) similarly as in the Bogoliubov coefficients but contrary to the Bogoliubov case the coefficients  $\alpha_k$  possess no singularity at  $k = 0$ .

Note that the coefficients (36) and (39) are proportional to  $V_0$ . Thus, for  $V_0 \ll 1$  (i.e. for the potential strength much smaller than the chemical potential of the system) the perturbation of the condensate wavefunction is certainly negligible. Deformation of the soliton shape depends on the relation of  $k_0$  to the inverse of the healing length. For  $k_0 \ll 1$ , i.e. if the potential

changes on a scale much larger than the healing length of the system, we may employ the Thomas-Fermi profile [30] and approximate the condensate wavefunction by  $\phi(x) \approx \sqrt{\rho_0[1 - V(x)]} \tanh(x/\tilde{\xi})$  where  $\tilde{\xi} = 1/\sqrt{1 + V_0}$  is a local healing length around the soliton position. Thus, for  $k_0 \ll 1$ , the soliton shape is still given by the hyperbolic tangent function but its size is smaller. In the other limit, i.e. for  $k_0 \gg 1$ , the condensate wavefunction reveals harmonic oscillations superposed on the hyperbolic tangent function. Indeed, for  $k_0 \gg 1$ , Eq. (39) implies that the dominant contribution comes from  $k \approx k_0$  which behaves like  $V_0/k_0^2$ . Thus, even if we increase  $V_0$  but choose  $k_0 \gg 1$  the wavefunction perturbation is still negligible due to smoothing effects. Finally we would like to stress that plotting  $\phi$  obtained in the Bogoliubov approach and with the help of the Pöschl-Teller modes one gets exactly the same results.

For comparison with numerics we have chosen the case of an optical speckle potential. Such a potential is characterized by: zero mean value  $\overline{V(x)} = 0$ , where the overbar denotes an ensemble average over disorder realizations, standard deviation  $V_0 = [\overline{V(x)^2}]^{1/2}$  and autocorrelation function  $\overline{V(x')V(x'+x)} = V_0^2 \frac{\sin^2(x/\sigma_R)}{(x/\sigma_R)^2}$  where  $\sigma_R$  is the correlation length of the disorder. In Fig. 1 we show examples of the solitonic solutions in the presence of an optical speckle potential for a correlation length much smaller ( $\sigma_R = 0.05$ ) and comparable ( $\sigma_R = 1$ ) to the healing length and for  $V_0 = 1$  and  $V_0 = 0.5$  respectively, obtained within the perturbation approaches and by a numerical solution of the GPE. The agreement is surprisingly good even though the strength of the disorder is of the order of the chemical potential. For  $\sigma_R = 0.05$  the disorder changes rapidly and its effect on the condensate is significantly smaller than for  $\sigma_R = 1$ .

We have considered a dark soliton in a box potential and analyzed its deformation due to the presence of a weak disorder potential. Our results can also be applied to the system in the presence of, e.g., a shallow harmonic trap. Indeed, if we are interested in the deformation of the condensate wavefunction in the vicinity of the trap center and if the change of the harmonic potential energy on a scale of the soliton size is much smaller than the disorder strength, i.e.  $\omega^2 \ll V_0$  where  $\omega$  is harmonic trap frequency, the effect of the presence of the trap on the soliton deformation can be neglected.

### III. QUANTUM DESCRIPTION

In the previous section, in order to describe ultra-cold atoms, we have applied the mean field approximation where it is assumed that a many body system is in a state where all atoms occupy the same single particle wavefunction. Solution of the GPE is an optimal choice for such a single particle wavefunction. Then, a stationary dark soliton appears as a solution of the classical wave equation and its position is given by a real number

$q$ . In the present section we will take into account situations when particles do not necessary occupy the same single particle state. It turns out that the problem can be described within the quantum version of the Bogoliubov approach where, e.g.,  $q$  becomes a quantum mechanical operator  $\hat{q}$  and the soliton position is described by a probability distribution. This approach has rather a semiclassical nature. The full quantum analysis would involve the N-body problem as has been done in Refs. [55, 56] for the bright solitons in optical fibres, see also Ref. [32].

#### A. Effective Hamiltonian

An effective Hamiltonian that describes a bright soliton in the presence of a weak external potential has been introduced in Ref. [36]. It is based on the Dziarmaga idea of how to describe non-perturbatively degrees of freedom corresponding to Bogoliubov zero modes [38]. Derivation of the effective Hamiltonian in the dark soliton case follows the same reasoning and therefore we will present key elements only.

In the previous section once the global phase  $\theta$  of the wavefunction (2) and the soliton position  $q$  have been chosen no deviations of them have been considered. Small deviations can be described with the help of the zero modes, compare (10)-(11) and (15), while large deviations need modifications of the description. The expansion of the wavefunction perturbation around a given value of the soliton position  $q$ , see (15), is actually not necessary because one may treat  $q$  as a dynamical variable and the same is true for  $\theta$  [38]. This way we obtain

$$\begin{aligned} \begin{bmatrix} \phi \\ \phi^* \end{bmatrix} &= \begin{bmatrix} \phi_0 \\ \phi_0^* \end{bmatrix} + P_\theta \begin{bmatrix} u_\theta^{ad} \\ v_\theta^{ad} \end{bmatrix} + P_q \begin{bmatrix} u_q^{ad} \\ v_q^{ad} \end{bmatrix} \\ &+ \sum_k \left( b_k \begin{bmatrix} u_k \\ v_k \end{bmatrix} + b_k^* \begin{bmatrix} v_k^* \\ u_k^* \end{bmatrix} \right). \end{aligned} \quad (41)$$

In (41) all modes depend on  $q$  and  $\theta$  and can follow large changes of the soliton position and global phase. Substituting (41) into the energy functional

$$H = \int dx \left[ \frac{1}{2} |\partial_x \phi|^2 + V |\phi|^2 + \frac{1}{2\rho_0} |\phi|^4 - \mu |\phi|^2 \right], \quad (42)$$

leads to

$$\begin{aligned} H &= -\frac{P_q^2}{2|M_q|} + \int dx V(x) |\phi_0(x - q)|^2 \\ &+ \frac{P_\theta^2}{2M_\theta} + 2P_\theta \langle u_\theta^{ad} | V \phi_0 \rangle \\ &+ \sum_k [\epsilon_k b_k^* b_k + s_k (b_k + b_k^*)], \end{aligned} \quad (43)$$

with

$$s_k = \langle u_k | V \phi_0 \rangle + \langle v_k | V \phi_0^* \rangle, \quad (44)$$

where only terms of the order  $\mathcal{O}(P^2, b^2, PV, bV)$  are kept. Note that Eq. (43) is the Hamiltonian formulation of the

classical perturbation theory applied in Sec.IIA. That is, fixed points of the Hamilton equations generated by (43) [36] correspond to stationary solutions analyzed in Sec.IIA. We know from Sec.II that such stationary solutions are in a very good agreement with the exact numerical calculations for the disorder strength even of the order of the chemical potential of the system, i.e. for  $V_0 \approx 1$ .

In the so-called second quantization formalism the quantum many body Hamiltonian corresponds to (42) where the wavefunction  $\phi$  is substituted by a bosonic field operator  $\hat{\phi}$ . Then, also the expansion coefficients in (41) become operators

$$\hat{P}_q = -i\partial_q, \quad (45)$$

$$\hat{P}_\theta = \hat{N} - N_0 = -i\partial_\theta, \quad (46)$$

and fulfill commutation relations

$$\begin{aligned} [\hat{q}, \hat{P}_q] &= i, \\ [\hat{\theta}, \hat{P}_\theta] &= i, \\ [\hat{b}_k, \hat{b}_{k'}^\dagger] &= \delta_{kk'}. \end{aligned} \quad (47)$$

Energy functional (43) does not depend on  $\theta$ , thus, in the quantum description  $[\hat{P}_\theta, \hat{H}] = 0$  and we may restrict it to the Hilbert subspace with exactly  $N_0$  particles, i.e. for any state in this subspace  $\hat{P}_\theta|\psi\rangle = 0$ , and the quantum effective Hamiltonian reduces to

$$\hat{H} = \hat{H}_q + \hat{H}_B + \hat{H}_1, \quad (48)$$

where

$$\begin{aligned} \hat{H}_q &= -\frac{\hat{P}_q^2}{2|M_q|} + \int dx V(x) |\phi_0(x-q)|^2 \\ &= -\left( \frac{\hat{P}_q^2}{2|M_q|} + \frac{|M_q|}{4} \int dx \frac{V(x)}{\cosh^2(x-q)} \right), \end{aligned} \quad (49)$$

$$\hat{H}_B = \sum_k \epsilon_k \hat{b}_k^\dagger \hat{b}_k, \quad (50)$$

$$\hat{H}_1 = \sum_k s_k (\hat{b}_k + \hat{b}_k^\dagger). \quad (51)$$

The Hamiltonian  $\hat{H}_q$  describes soliton motion in an effective potential which turns out to be a convolution of the original potential with the density  $|\phi_0|^2$ . Owing to  $|\phi_0|^2 = \rho_0 \tanh^2(x-q) = \frac{|M_q|}{4} [1 - \cosh^2(x-q)]$  and  $\overline{V(x)} = 0$ ,  $\hat{H}_q$  becomes similar to the corresponding Hamiltonian for a bright soliton in a weak external potential [35]. The term  $\hat{H}_B$  describes quasi-particle subsystem (phonons) and  $\hat{H}_1$  is the part of the Hamiltonian that couples soliton position degree of freedom with phonons. In the classical description (Sec. II) such a coupling is responsible for the deformation of the stationary condensate wavefunction. In the following we will not look for eigenstates of the total Hamiltonian  $\hat{H}$  but rather consider eigenstates of  $\hat{H}_q$  and calculate the lifetime of the

system prepared in these eigenstates due to the coupling with the quasi-particle subsystem induced by  $\hat{H}_1$ .

We would like to emphasize striking differences between the classical description and the present quantum description. They are most apparent in the absence of an external potential. Indeed, for  $V(x) = 0$ , the soliton position in the classical description can be chosen arbitrary but it is well defined. In the quantum approach the Hamiltonian  $\hat{H}_q$  tells us that eigenstates of the system correspond to eigenstates of the momentum operator  $\hat{P}_q$  and the corresponding probability distributions for the soliton position are totally delocalized. Thus, similarly as in the bright soliton case [32], dramatic condensate fragmentations are predicted in the quantum approach of a dark soliton [57].

## B. Anderson localization of a dark soliton

The final form of the Hamiltonian  $\hat{H}_q$  is similar to the Hamiltonian for the center of mass of a bright soliton in a weak external potential [36]. The effective mass  $|M_q|$  in (49) is equal to two times the number of particles missing in a dark soliton notch while in the bright soliton case it is given by a total number of particles in a system. A bright soliton is the ground state solution of the GPE and excitation of its center of mass increases the energy of the system. A dark soliton corresponds to a collectively excited system and in order to decrease the system energy one has to, e.g., accelerate the soliton. Indeed, excitations of the soliton position degree of freedom actually decrease the system energy due to the minus sign in front of the expression (49).

It has been shown that in the presence of a weak disorder potential the center of mass of a bright soliton reveals Anderson localization [35]. The same phenomenon can be expected for dark solitons. For  $V(x)$  being an optical speckle potential with the correlation length  $\sigma_R$  smaller than the healing length of the system we obtain the effective potential in (49) where the healing length plays a role of the effective correlation length. The generic properties of the Anderson localization in 1D allow us to expect that all eigenstates of the Hamiltonian  $\hat{H}_q$  are exponentially localized, i.e., have a shape with the overall envelope [58, 59]

$$|\psi_n(q)|^2 \propto \exp\left(-\frac{|q - q_0|}{l_{loc}}\right), \quad (52)$$

where  $\hat{H}_q \psi_n(q) = E_n \psi_n(q)$ ,  $q_0$  is the mean position of the soliton and  $l_{loc} = l_{loc}(E_n)$  is the localization length. Indeed, in Fig. 2 we present examples of the Anderson localized eigenstates for two values of the standard deviation of the speckle potential  $V_0$ . The parameters we have chosen correspond to:  $N_0 = 10^5$   $^{87}\text{Rb}$  atoms in a quasi-1D box potential of  $L = 3550$  (3.37 mm) with the harmonic potential of  $\omega_\perp = 2\pi \times 370$  Hz in the transverse directions; the correlation length of the speckle potential

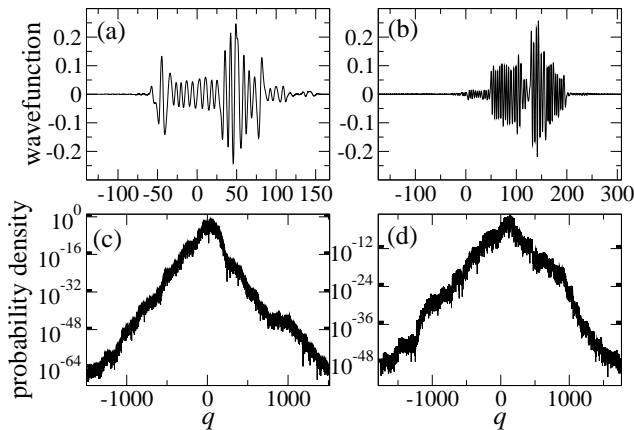


Figure 2: In top panels we show examples of eigenstates of the effective Hamiltonian  $\hat{H}_q$ , see (49), while in bottom panels the corresponding probability densities in log scale. The correlation length of the speckle potential  $\sigma_R = 0.28$  and the strength  $V_0 = 7 \times 10^{-5}$  (left panels) and  $V_0 = 1.4 \times 10^{-4}$  (right panels). The eigenstates correspond to the eigenvalue  $E_n = -3.03 \times 10^{-3}$  (left panels) and  $E_n = -8.58 \times 10^{-3}$  (right panels) and reveal the localization length  $l_{loc} = 10.5$  and  $l_{loc} = 15.7$ , respectively.

$\sigma_R = 0.28$  ( $0.27 \mu\text{m}$ ) [16]. The energy units (3) are the following:  $E_0/\hbar = 789 \text{ Hz}$ ,  $l_0 = 0.96 \mu\text{m}$  and  $t_0 = 1.27 \text{ ms}$ .

In order to obtain predictions for Anderson localization of solitons we have neglected coupling of the soliton position degree of freedom to the quasi-particle subsystem. In the bright soliton case such an approximation is justified because there is a huge energy gap for quasi-particle excitations and if the strength of the potential is much smaller than the chemical potential of the system, corrections to the effective Hamiltonian  $\hat{H}_q$  are negligible [36]. In the dark soliton case there is practically no energy gap for phonon excitations, i.e. minimal  $\epsilon_k$ , see (9), corresponds to  $k = \pi/L$  which tends to zero for a large system. Moreover, a dark soliton is a collectively excited state which may decay to lower energy states by *emission* of phonons. If the strength of the disorder potential  $V_0 \ll 1$  we know from the classical analysis that the shape of a stationary dark solution of the GPE is negligibly deformed by the external potential. In the quantum description we may thus expect that the lifetime of Anderson localized eigenstates is sufficiently long and allows for experimental observations of the localization effects.

Suppose we choose an initial state  $|\Psi\rangle$  of the  $N_0$ -particle system where the soliton position is described by an eigenstate  $\psi_n(q)$  of the Hamiltonian  $\hat{H}_q$  corresponding to an eigenvalue  $E_n$  and there is no phonon excitation, i.e. we deal with the quasi-particle vacuum state,

$$|\Psi\rangle = |\psi_n, 0_B\rangle = \psi_n(q)|0_B\rangle, \quad (53)$$

where  $\hat{b}_k|0_B\rangle = 0$  for each  $k$ . In the first order in the  $\hat{H}_1$ , see (51), the system may decay to another eigenstate  $\psi_m(q)$  corresponding to an eigenvalue  $E_m$  emitting a single phonon of energy  $\epsilon_k$ . According to the Fermi

golden rule the decay rate reads

$$\Gamma = 2\pi \sum_m \gamma_m, \quad (54)$$

where

$$\gamma_m = |\langle \psi_m, 1_k | \hat{H}_1 | \psi_n, 0_B \rangle|^2 g(\epsilon_k) = |\langle \psi_m | s_k | \psi_n \rangle| g(\epsilon_k). \quad (55)$$

The sum in (54) runs over all eigenstates  $\psi_m$  for which we can find such a phonon that the energy conservation  $E_n = E_m + \epsilon_k$  is fulfilled. We assume a continuum phonon spectrum (9) with the energy gap corresponding to  $k = \pi/L$ . The density of states is

$$g(\epsilon) = \frac{\epsilon}{[2(\epsilon^2 + 1)(\sqrt{\epsilon^2 + 1} - 1)]^{1/2}}. \quad (56)$$

The lifetime of the Anderson localized states presented in Figs. 2a and 2c  $\tau = 1/\Gamma = 8 \times 10^5$  (17 minutes) and in Figs. 2b and 2d  $\tau = 2.5 \times 10^5$  (5 minutes) which means that there is by far enough time to perform experiments until they can decay due to phonon emissions. In Fig. 3 we show contributions  $\gamma_m$  to the decay rate (54) and  $\psi_m$  states corresponding to the largest values of  $\gamma_m$ . Figure 3 indicates that the most probable decay leads the system to states localized in the vicinity of the initial localization region.

Long lifetime of Anderson localized states is very promising from the experimental point of view. Indeed, it means that there is sufficient time to excite a dark soliton in an ultra-cold atomic gas, wait until it localizes in the presence of a weak disorder potential and perform an atom density measurement. If the soliton is Anderson localized distribution of soliton positions collected in many realizations of the experiment [57] will reveal an exponential profile.

Experiments with dark solitons have been performed in the presence of a harmonic trap [6, 60–64]. In such a case, in order to observe the Anderson localization of the solitons, the trap has to be sufficiently shallow. That is, the ground state extension of the soliton position in the harmonic trap without the disorder must be much larger than the localization length predicted in our analysis, i.e.  $\frac{1}{\sqrt{|M_q|\omega}} \gg l_{loc}$  where  $\omega$  is the harmonic trap frequency.

#### IV. CONCLUSIONS

We have considered a dark soliton in dilute ultra-cold atomic gases in the presence of a weak disorder potential. Our consideration is divided into classical and quantum descriptions. The classical approach concerns the analysis of stationary solutions of the Gross-Pitaevskii equation and the effect of deformation of the soliton shape by the disorder. We have employed two methods: the Bogoliubov approach and the expansion of a wavefunction perturbation in eigenmodes of the Pöschl-Teller potential. These two methods lead to the same results,



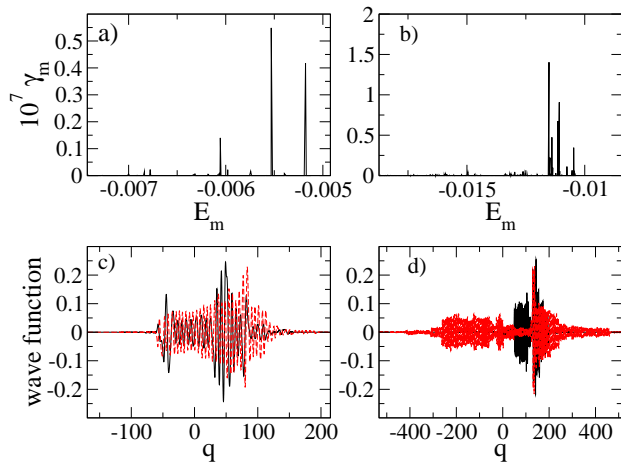


Figure 3: (Color online) Top panels: contributions  $\gamma_m$ , Eq. (55), to the total decay rate  $\Gamma$  as a functions of energy  $E_m$ . Bottom panels: initial states  $\psi_n(q)$  of the system (solid black lines) and eigenstates  $\psi_m(q)$  (dashed red lines) corresponding to the most probable decay channels, i.e. maximal values of  $\gamma_m$ . Parameters chosen in left (right) panels are the same as in the corresponding panels in Fig. 2.

however, the expansion in the Pöschl-Teller modes turns out to be more convenient and in particular allows us to obtain a very simple form of the soliton perturbation in terms of an integral kernel. Comparison of the perturbative calculations with the numerical results shows surprisingly good agreement even for the strength of an

external potential as great as the chemical potential of the system. If the strength is much smaller than the chemical potential the wavefunction deformation is negligibly small.

The Bogoliubov approach is invaluable in the quantum description where we are interested in many body eigenstates of the system. If the strength of an external potential is much smaller than the chemical potential the dark soliton position may be described by an effective quantum Hamiltonian which is weakly coupled to the quasi-particle subsystem. The effective Hamiltonian turns out to be similar to the corresponding Hamiltonian in the problem of a bright soliton in a weak external potential. Similarly as in the bright soliton case we predict Anderson localization of a dark soliton in the presence of a disorder potential. Because there is a coupling between the soliton position degree of freedom and the quasi-particle subsystem the localized states may decay via phonon emission process. We have investigated lifetimes of the Anderson localized states and it turns out that for typical experimental conditions they exceed condensate lifetimes that make experimental observations of the dark soliton localization realistic.

#### Acknowledgments

This work is supported by the Polish Government within research project 2009-2012 (MM) and by National Science Centre under projects DEC-2011/01/N/ST2/00418 (MP) and DEC-2011/01/B/ST3/00512 (KS).

- 
- [1] I. Bloch, J. Dalibard, W. Zwerger, Rev. Mod. Phys. **80**(3),885 (2008).
  - [2] D. Jaksch, P. Zoller, Ann. Phys. **315**, 52 (2005).
  - [3] I. Buluta, F. Nori, Science **326**, 108 (2009).
  - [4] C. A. Regal, M. Greiner, and D. S. Jin, Phys. Rev. Lett. **92**, 040403 (2004).
  - [5] M. Inguscio, W. Ketterle, and C. Salomon (Editors), *Ultra-cold Fermi Gases*, Proceedings of the International School of Physics Enrico Fermi, Course CLXIV, Varenna 2006, (IOS Press, Amsterdam) 2007.
  - [6] S. Burger, K. Bongs, S. Dettmer, W. Ertmer, K. Sengstock, A. Sanpera, G. V. Shlyapnikov, M. Lewenstein, Phys. Rev. Lett. **83**, 5198 (1999).
  - [7] J. Denschlag, J. E. Simsarian, D. L. Feder, Charles W. Clark, L. A. Collins, J. Cubizolles, L. Deng, E. W. Hagley, K. Helmerson, W. P. Reinhardt, S. L. Rolston, B. I. Schneider and W. D. Phillips, Science **287**, 97 (2000).
  - [8] L. Khaykovich, F. Schreck, G. Ferrari, T. Bourdel, J. Cubizolles, L. D. Carr, Y. Castin, C. Salomon, Science **296**, 1290 (2002).
  - [9] K. E. Strecker, G. B. Partridge, A. G. Truscott, R. G. Hulet, Nature **417**, 150 (2002).
  - [10] M. R. Matthews, B. P. Anderson, P. C. Haljan, D. S. Hall, C. E. Wieman, E. A. Cornell Phys. Rev. Lett. **83**, 2498 (1999).
  - [11] K. W. Madison, F. Chevy, W. Wohlleben, J. Dalibard Phys. Rev.Lett. **84**, 806 (2000).
  - [12] J. R. Abo-Shaeer, C. Raman, J. M. Vogels, W. Ketterle Science **292**, 476 (2001).
  - [13] D. Jaksch, C. Bruder, J. I. Cirac, C. W. Gardiner, P. Zoller, Phys. Rev. Lett. **81**, 3108 (1998).
  - [14] M. P. A. Fisher, P. B. Weichman, G. Grinstein, D. S. Fisher, Phys. Rev. B **40**, 546 (1989).
  - [15] M. Greiner, O. Mandel, T. Esslinger, T. W. Hänsch, and I. Bloch, Nature **415**, 39 (2002).
  - [16] J. Billy, V. Josse, Z. Zuo, A. Bernard, B. Hambrecht, P. Lugan, D. Clement, L. Sanchez-Palencia, P. Bouyer, and A. Aspect, Nature **453**, 891 (2008).
  - [17] G. Roati, C. D'Errico, L. Fallani, M. Fattori, C. Fort, M. Zaccanti, G. Modugno, M. Modugno, and M. Inguscio, Nature **453**, 895 (2008).
  - [18] S. S. Kondov, W. R. McGehee, J. J. Zirbel, B. DeMarco, Science **334**, 66 (2011).
  - [19] F. Jendrzejewski, A. Bernard, K. Müller, P. Cheinet, V. Josse, M. Piraud, L. Pezze, L. Sanchez-Palencia, A. Aspect, and P. Bouyer, arXiv:1108.0137.
  - [20] Y.-J. Lin, R. L. Compton, A. R. Perry, W. D. Phillips, J. V. Porto, I. B. Spielman, Phys. Rev. Lett. **102**, 130401 (2009).
  - [21] Yu-Ju Lin, Rob L. Compton, Karina J. Garcia, James V. Porto, Ian B. Spielman, Nature **462**, 628 (2009).
  - [22] J. Dalibard, F. Gerbier, G. Juzeliunas, P. Öhberg,

- arXiv:1008.5378.
- [23] P. W. Anderson, Phys. Rev. **109**, 1492 (1958).
  - [24] B. Damski, J. Zakrzewski, L. Santos, P. Zoller, M. Lewenstein, Phys. Rev. Lett. **91**, 080403 (2003).
  - [25] J. E. Lye, L. Fallani, M. Modugno, D. Wiersma, C. Fort, M. Inguscio, Phys. Rev. Lett. **95**, 070401 (2005).
  - [26] C. Fort, L. Fallani, V. Guarrera, J. Lye, M. Modugno, D. Wiersma, M. Inguscio, Phys. Rev. Lett. **95**, 170410 (2005).
  - [27] D. Clément *et al.*, Phys. Rev. Lett. **95**, 170409 (2005).
  - [28] T. Schulte, S. Drenkelforth, J. Kruse, W. Ertmer, J. Arlt, K. Sacha, J. Zakrzewski, M. Lewenstein, Phys. Rev. Lett. **95**, 170411 (2005).
  - [29] T. Schulte, S. Drenkelforth, J. Kruse, R. Tiemeyer, K. Sacha, J. Zakrzewski, M. Lewenstein, W. Ertmer, J. J. Arlt, New J. Phys. **8**, 230 (2005).
  - [30] L.P. Pitaevskii, Sov. Phys. JETP **13**, 451 (1961); E.P. Gross, Nuovo Cimento **20**, 454 (1961).
  - [31] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999).
  - [32] Y. Castin, in Les Houches Session LXXII, *Coherent atomic matter waves 1999*, edited by R. Kaiser, C. Westbrook and F. David, (Springer-Verlag Berlin Heidelberg New York 2001).
  - [33] C. Weiss, Y. Castin, Phys. Rev. Lett. **102**, 010403 (2006).
  - [34] M. Lewenstein, B. A. Malomed, New. J. Phys. **11**, 113014 (2009).
  - [35] K. Sacha, C. A. Müller, D. Delande, J. Zakrzewski, Phys. Rev. Lett. **103**, 210402 (2009).
  - [36] K. Sacha, D. Delande, J. Zakrzewski, Acta Physica Polonica A **116**, 772 (2009).
  - [37] T. P. Meyrath, F. Schreck, J. L. Hanssen, C.-S. Chuu, and M. G. Raizen, Phys. Rev. A **71**, 041604R (2005).
  - [38] J. Dziarmaga, Phys. Rev. A **70**, 063616 (2004).
  - [39] D. Clement, A F Varon, J A Retter, L Sanchez-Palencia, A Aspect, and P Bouyer, New J. Phys. **8**, 165 (2006).
  - [40] M. Płodzień, and K. Sacha, Phys. Rev. A **84**, 023624 (2011); M. Piraud, A. Aspect, L. Sanchez-Palencia, arXiv:1104.2314.
  - [41] C. Müller, Appl. Phys. B **102**, 459 (2011).
  - [42] S.A. Gredeskul, Y.S. Kivshar, Phys. Rep. **216**, 1 (1992).
  - [43] N. Bilas and N. Pavloff, Phys. Rev. Lett. **95**, 130403 (2005).
  - [44] Y. S. Kivshar and B. Luther-Davies, Physics Reports **298**, 81 (1998).
  - [45] D. J. Frantzeskakis, J. Phys. A **43**, 213001 (2010).
  - [46] S. Giorgini, L. Pitaevskii, S. Stringari, Phys. Rev. B **49**, 12938 (1994).
  - [47] L. Sanchez-Palencia, Phys. Rev. A **74**, 053625 (2006).
  - [48] C. Gaul, N. Renner, C. A. Müller, Phys. Rev. A **80**, 053620 (2009).
  - [49] C. Gaul, C. A. Müller, Phys. Rev. A **83**, 063629 (2011).
  - [50] M. Lewenstein, L. You, Phys. Rev. Lett. **77**, 3489 (1996).
  - [51] Y. Castin, R. Dum, Phys. Rev. A **57**, 2008 (1998).
  - [52] D. E. Pelinovsky and P. G. Kevrekidis, Z. angew. Math. Phys. **59**, 559 (2008).
  - [53] G. Pöschl, E. Teller, Z. Phys. **83**, 143 (1933).
  - [54] J. Lekner, Am. J. Phys. **75**, 1151 (2007).
  - [55] Y. Lai and H. A. Haus, Phys. Rev. A **40**, 844 (1989).
  - [56] Y. Lai and H. A. Haus, Phys. Rev. A **40**, 854 (1989).
  - [57] J. Dziarmaga and K. Sacha, J. Phys. B, **39**, **43** (2006); R. V. Mishmash, I. Danshita, C. W. Clark, and L. D. Carr, Phys. Rev. A **80**, 053612 (2009); J. Dziarmaga, P. Deuar, and K. Sacha, Phys. Rev. Lett. **105**, 018903 (2010); R. V. Mishmash, and L. D. Carr, Phys. Rev. Lett. **105**, 018904 (2010).
  - [58] I.M. Lifshitz, S.A. Gredeskul, L.A. Pastur, *Introduction to the Theory of Disordered Systems* (Wiley, New York, 1988).
  - [59] B. van Tiggelen, in *Diffuse Waves in Complex Media*, edited by J.-P. Fouque, NATO Advanced Study Institutes, Ser. C, Vol. 531 (Kluwer, Dordrecht, 1999).
  - [60] B. P. Anderson<sup>1</sup>, P. C. Haljan, C. A. Regal, D. L. Feder, L. A. Collins, C. W. Clark, and E. A. Cornell, Phys. Rev. Lett. **86**, 2926 (2001).
  - [61] C. Becker, S. Stellmer, P. Soltan-Panahi, S. Dörscher, M. Baumert, E.-M. Richter, J. Kronjäger, K. Bongs, and K. Sengstock, Nature Physics **4**, 496 (2008).
  - [62] S. Stellmer, C. Becker, P. Soltan-Panahi, E.-M. Richter, S. Dörscher, M. Baumert, J. Kronjäger, K. Bongs, and K. Sengstock, Phys. Rev. Lett. **101**, 120406 (2008).
  - [63] A. Weller, J. P. Ronzheimer, C. Gross, J. Esteve, and M. K. Oberthaler, D. J. Frantzeskakis, G. Theocharis and P. G. Kevrekidis, Phys. Rev. Lett. **101**, 130401 (2008).
  - [64] I. Shomroni, E. Lahoud, S. Levy, J. Steinhauer, Nature Physics **5**, 193 (2009).